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Perimeter Bethe ansatz

R J Baxter

Department of Theoretical Physics, Research School of Physical Sciences, The Australian National University, GPO Box 4, Canberra, ACT 2601, Australia

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Abstract. The partition function of a finite Z-invariant six-vertex lattice model (with a prescribed arrow configuration at the boundary) is given. The expression is of the same type that occurs in the Bethe ansatz, but there are no 'wavenumber' equations to solve.

1. Introduction

Lieb (1967a, b) and Sutherland (1967) solved the regular square lattice zero-field six-vertex model, in that they obtained explicit solutions for the partition function per site $Z^{1/N}$ in the thermodynamic large-lattice limit. Their results have been generalised to a planar 'Z-invariant' eight-vertex model (Baxter 1978).

Special cases of the six- and eight-vertex models are the Ising (Onsager 1944) and free-fermion (Fan and Wu 1970) models. For a lattice of N sites these models have the property that the partition function Z can be expressed in terms of Pfaffians, and hence in terms of $4N$ by $4N$ (or $2N$ by $2N$) determinants. For a regular (i.e. homogeneous) lattice these determinants are cyclic, so one can express Z in terms of explicit N-fold products. The passage to the infinite-lattice limit is then straightforward.

The situation is nothing like so satisfactory for the general six- and eight-vertex models. There one can write down the Bethe ansatz for eigenvectors and eigenvalues of the transfer matrix. However, for a square lattice of $2n$ columns, for each eigenvalue one has to solve n non-linear equations in n unknowns (the 'wavenumbers' $k_1, \ldots, k_n$). This can only be done explicitly in the large-n limit, and even then the eigenvalue spectrum has not been obtained explicitly. (It involves solving a system of non-linear integral equations: Yang and Yang 1969, Gaudin 1971, Johnson and McCoy 1972, Takahashi and Suzuki 1972, Fowler 1982, Zotos 1982.) Usually one can solve explicitly only for the largest (and near-largest) eigenvalue, which limits one to considering the full ∞ by ∞ square lattice. One disadvantage of this is that the necessary analysis depends on the values of the input parameters, namely the Boltzmann weights of the model. For positive real weights there seems to be no problem, but for negative or complex weights difficulties can arise (in particular, in determining the contributing eigenvalue of largest modulus). Such complex weights can arise when mapping one model to another (e.g. the Potts to the six-vertex model: Baxter *et al* 1976), and the partition function may then be sensitive to boundary conditions (Baxter 1982a, 1986).

The purpose of this paper is to show that for a general finite Z-invariant six-vertex model, the partition function is given explicitly (apart from a simple factor) by the Bethe ansatz expression (3.2). However, in this case the $u_1, \ldots, u_n$ (which are related to the 'wavenumbers' $k_1, \ldots, k_n$) do not need to be evaluated from a complicated set of simultaneous equations: they are known explicitly.

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For a lattice of n lines the expression (3.2) is a sum of $n!$ terms, so it is still far from being immediately useful. Further, some delicate limits need to be taken in order to return to the homogeneous square lattice. Even so, there has been some success recently in handling Bethe ansatz expressions (Gaudin *et al* 1981, Korepin 1982, Gaudin 1983, appendix J): some special cases of the general result of this paper are implicitly contained in the second reference. The author's hope is that it may be possible to apply similar techniques to (3.2), so as to reduce it to a tractable expression, perhaps a determinant.

2. Z-invariant six-vertex model

As in Baxter (1978), consider some simply connected convex planar domain $\mathscr{D}$, such as the interior of a circle, and draw n straight lines within it, starting and ending at the boundary (perimeter) of $\mathscr{D}$. No three lines are allowed to intersect at a common point.

The intersections of these lines form the sites of a graph, or 'lattice', $\mathscr{L}$. The line segments between sites form the edges of $\mathscr{L}$. Each site is the end-point of four edges.

A typical graph $\mathscr{L}$ is shown in figure 1. Going anti-clockwise round the boundary from some point B, label the end-points of the lines of $\mathscr{L}$ as $1, 2, \ldots, 2n$, as in figure 1. If a line has end-points i and j, then we shall refer to it as 'the line (i, j)' where $1 \leqslant i < j \leqslant 2n$.

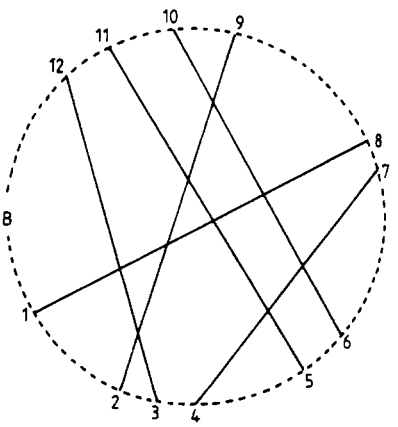

Figure 1. A typical graph $\mathscr{L}$, showing the labelling of the end-points.

Let λ be a given constant (a complex number), and with each end-point i associate a 'rapidity' v_i. For each line (i, j) impose the constraint

$$v_j = v_i + \lambda. \tag{2.1}$$

We shall call v_i the 'left' rapidity of the line (i, j), and v_j the 'right' rapidity (having in mind the orientation of figure 2). We shall also take $V = \{v_1, \ldots, v_{2n}\}$ to be the set of all $2n$ rapidities, V_l to be the set of n left-rapidities, and V_r the set of n right-rapidities. Thus $V = V_l \cup V_r$.

Two functions that we shall use are

$$f(v) = \sinh(v)/\sinh(\lambda - v) \qquad g(v) = \sinh(\lambda)/\sinh(\lambda - v). \tag{2.2}$$

Consider two lines (i, j) and (k, l), where $i < k$. There are three possible orderings of $i, j, k\, l$, and the corresponding line arrangements (distorted so as to flatten the boundary) are shown in figure 2. Any site of $\mathscr{L}$ is the intersection of two lines (i, j) and (k, l), so can be drawn as in figure 2(a), and referred to as site $(i, j|k, l)$.

Now construct a six-vertex model on $\mathscr{L}$. Place arrows on the edges of $\mathscr{L}$ so that at each site there are two arrows in and two arrows out (the 'ice-rule'). Then there are six allowed arrangements of arrows at a site $(i, j|k, l)$ of $\mathscr{L}$, as shown in figure 3. With the six arrangements associate weights $\omega_1, \ldots, \omega_6$, where

$$\omega_1 = \omega_2 = 1 \qquad \omega_3 = \omega_4 = f(v_k - v_i) \qquad \omega_5 = \omega_6 = g(v_k - v_i). \tag{2.3}$$

In addition, if the pair of lines (i, j) and (k, l) do not intersect in $\mathscr{D}$, it is convenient to associate with them a weight Ω_b if they are in configuration b of figure 2 (Ω_c if in configuration c), where

$$\Omega_b = f(v_k - v_l) \qquad \Omega_c = 1. \tag{2.4}$$

These weights Ω_b, Ω_c are independent of the arrow directions on the lines, so trivially modify the partition function.

The weights (2.3)-(2.4) are all real if $\lambda, u_1, \ldots, u_n$ are either all real, or all pure imaginary.

We use fixed boundary conditions, in which we specify the arrows on the $2n$ boundary edges. Because of the ice rule, n of these must point into $\mathscr{D}$, and n out. Thus we can specify the boundary arrows by giving the end-point locations $x_1, \ldots, x_n$ of the out-pointing arrows, where

$$1 \leqslant x_1 < x_2 < \ldots < x_n \leqslant 2n. \tag{2.5}$$

The partition function Z is then a function of $X = \{x_1, \ldots, x_n\}$ and $V = \{v_1, \ldots, v_{2n}\}$, so we can write it as $Z(X \mid V)$. It is defined by

$$Z(X \mid V) = \sum_{C} \prod_{(i,j|k,l)} (\text{weights}) \tag{2.6}$$

where the sum is over all allowed arrangements of arrows on the internal edges of $\mathscr{L}$, and the product is over all the $n\ (n-1)/2$ pairs (i, j), (k, l) of lines, each pair being given the appropriate weight determined by (2.3) and (2.4).

Thus the product in (2.6) is over all sites of $\mathscr{L}$, but also includes the arrow-independent weights (2.4).

This model is Z-invariant (Baxter 1978), which means that any triplet of lines satisfies the 'triangle' (or Young-Baxter) equations. This implies that $Z(X \mid V)$ is unchanged by shifting the lines of $\mathscr{L}$, provided their order at the boundary (and of course the boundary arrow arrangement) is kept fixed.

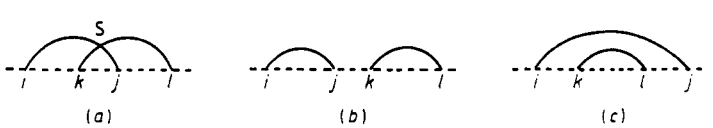

Figure 2. The three possible arrangements of two lines (i, j) and (k, l).

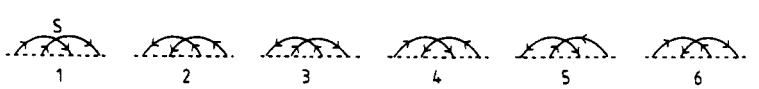

Figure 3. The six possible configurations of arrows at a site S of $\mathscr{L}$: S may be deep inside $\mathscr{L}$, the arrows being on the neighbouring four edges of S.

2.1. Recursion relation

We can consider the effect of passing the end-points of two different lines through one another. This will give us recursion relations for $Z(X \mid V)$.

At first sight there are six cases to consider: interchanging i with k, k with j, or j with l in figure 2(a), j with k in figure 2(b); or i with k or l with j in figure 2(c). However, the last three are the results of the first, so we need only consider interchanges of two consecutive end-points in figure 2(a): i with k, k with j, or j with l.

Whichever pair we select, they must be adjacent end-points of $\mathscr{L}$: let them be m, $m+1$ (where $1 \leqslant m < 2n$). Thus the three cases are $i, k = m, m+1$; $k, j = m, m+1$; and $j, l = m, m+1$. The corresponding end-point rapidities are v_m, v_{m+1}.

As the two end-points come together, the site S formed by the intersection of the two lines approaches them: we obtain the picture in figure 4, which also shows the six possible arrow arrangements round the site.

Figure 4 is of course just a distortion of figure 3, but the distortion is different for the three cases. Let $\omega_1', \ldots, \omega_6'$ be the weights of the six arrangements in figure 4. They can be obtained by translating back to figure 3 and using (2.3). It is remarkable that *for all three cases*

$$\begin{aligned} &\omega_1' = \omega_2' = \xi \qquad \omega_3' = \omega_4' = \xi f(v_{m+1} - v_m) \\ &\omega_5' = \omega_6' = \xi g(v_{m+1} - v_m). \end{aligned} \tag{2.7}$$

Here ξ is a normalisation factor of $\omega_1', \ldots, \omega_6'$ that does depend on the case under consideration: for the second case $(k, j = m, m+1)\xi = \Omega_b$; otherwise $\xi = \Omega_c = 1$.

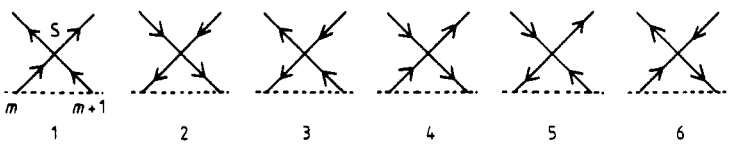

Figure 4. A site S close to the boundary, showing the six arrow configurations.

As the two end-points pass through one another, they reverse their sequence and the site S moves outside $\mathscr{D}$. In the first and third cases the resulting line configuration is that of figure 2(c), while in the second case it is that of figure 2(b) (with the same labels i, j, k, l). Thus ξ is just the weight of the new disjoint line configuration.

Passing the end-points through one another is equivalent to raising the boundary in figure 4 through the site above it. Fixing the lower two arrows and considering the possible arrangements of the upper two, it follows that

$$\begin{aligned} &Z(\ldots \mid V) = Z(\ldots \mid V') \\ &Z(\ldots m \ldots \mid V) = g(v_{m+1} - v_m) Z(\ldots m \ldots \mid V') + f(v_{m+1} - v_m) Z(\ldots m+1 \ldots \mid V') \\ &Z(\ldots m+1 \ldots \mid V) = f(v_{m+1} - v_m) Z(\ldots m \ldots \mid V') + g(v_{m+1} - v_m) Z(\ldots m+1 \ldots \mid V') \\ &Z(\ldots m, m+1 \ldots \mid V) = Z(\ldots m, m+1 \ldots \mid V'). \end{aligned} \tag{2.8}$$

Here we have used some shorthand notation: $V = \{v_1, \ldots, v_{2n}\}$ and V' is V with v_m and v_{m+1} interchanged. Dots refer to locations x_j that are not equal to m or $m+1$,

and are given the same meaning in all terms within an equation. Thus ... in the first equation means $x_1, \ldots, x_n$, with no $x_j = m$ or $m+1$: throughout the second and third equations $\ldots p \ldots$ (with $p = m$ or $m+1$) means $x_1, \ldots, x_{r-1}, p, x_{r+1}, \ldots, x_n$ (for $1 \leq r \leq n$, $x_{r-1} < m$, $x_{r+1} > m+1$); and in the fourth equation $\ldots m, m+1 \ldots$ means $x_1, \ldots, x_{r-1}, m, m+1, x_{r+2}, \ldots, x_n$.

We can regard $Z(X \mid V)$ as the element X of a vector $z(V)$. Since the equations (2.8) are linear, they can be written as

$$z(V) = A(v_{m+1} - v_m) z(V') \tag{2.9}$$

where $A(v_{m+1} - v_m)$ is the matrix of coefficients of the RHS: with an appropriate ordering of elements, it breaks up into one-by-one and two-by-two diagonal blocks. It satisfies the relation

$$A(v)A(-v) = \mathbb{1} \tag{2.10}$$

so (2.9) can also be written as

$$z(V') = A(v_m - v_{m+1}) z(V). \tag{2.11}$$

This is just (2.9) with v_m interchanged with v_{m+1}, so (2.8) remains true if v_m and v_{m+1} are interchanged.

It follows that (2.8) is true regardless of whether or not the end-points m and $m+1$ belong to lines that intersect inside $\mathcal{D}$, so long as the lines are distinct (i.e. m and $m+1$ must not be end-points of the same line).

2.2. *End condition*

We can in principle make repeated use of (2.8) to relate $Z(X \mid V)$ to the partition function of the trivial graph shown in figure 5, where

$$v_{2n+1-j} = v_j + \lambda \qquad j = 1, \ldots, n. \tag{2.12a}$$

Denote V in this case as V_0. For every pair $(j, 2n+1-j)$ of end-points (where $j = 1, \ldots, n$), there must now be just one out-pointing arrow. Setting $y_r = \min(x_r, 2n+1-x_r)$ and noting that all the line pairs are in configuration 2_c and hence

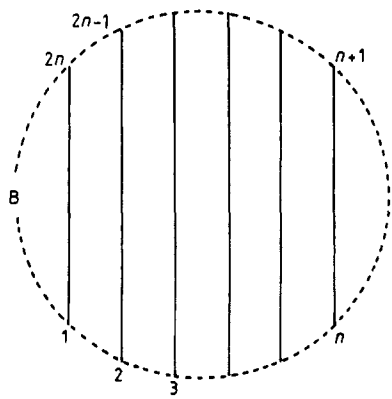

Figure 5. A trivial graph of n non-intersecting straight lines.

have weight $\Omega_c = 1$, it follows that for the graph in figure 5 (for all $x_1, \ldots, 4x_n$ satisfying (2.5)),

$$Z(X|V_0) = 1 \qquad \text{if } \{y_1, \ldots, y_n\} \text{ is a permutation of } \{1, \ldots, n\}$$
$$= 0 \qquad \text{otherwise.} \tag{2.12b}$$

For any straight-line graph $\mathscr{L}$, $Z(X|V)$ is defined by (2.8) and (2.12). We can think of (2.12) as an 'end condition' supplementing the recursion relation (2.8).

2.3. Triangle relations

Incidentally, the triangle relations take a particularly simple form in terms of the recursion matrix $A(v)$ (which can be thought of as a 'vertex transfer matrix'). They are

$$A(v-u)A(w-u)A(w-v) = A(w-v)A(w-u)A(v-u) \tag{2.13}$$

for all complex numbers u, v, w. (This is equivalent to the six-vertex model case of equation (9.7.14) of Baxter 1982b.)

Because of this property, $Z(X|V)$ is independent of the ordering of the (v_m, v_{m+1}) interchanges used to obtain it, via (2.8), from $Z(X|V_0)$. This is equivalent to saying that the model is Z-invariant.

3. Bethe ansatz

Define a 'single-particle' or 'single out-arrow' function

$$\phi(u, x) = \prod_{j=1}^{x-1} \sinh(u - v_j + \lambda) \times \prod_{j=x+1}^{2n} \sinh(v_j - u) \tag{3.1}$$

where $1 \leqslant x \leqslant 2n$ and u is an arbitrary complex number. Obviously this depends also on the rapidities $V = \{v_1, \ldots, v_{2n}\}$. Further define

$$f(X|V|U) = \sum_P \left\{ \prod_{1 \leqslant i < j \leqslant n} \sinh(u_i - u_j + \lambda)/\sinh(u_j - u_i) \right\}$$
$$\times \phi(u_1, x_1)\phi(u_2, x_2) \ldots \phi(u_n, x_n). \tag{3.2}$$

Here $U = \{u_1, \ldots, u_n\}$ is some given set of complex numbers, as yet arbitrary. The summation is over all permutations P of $u_1, \ldots, u_n$.

3.1. Recursion relation

For all values of U, this function $f(X|V|U)$ satisfies the same recurrence relations (2.8) as $Z(X|V)$. The first of these relations can be established by noting that each function $\phi(u, x)$ is a symmetric function of v_m and v_{m+1}, provided $x \neq m$ or $m+1$. The second and third then follow from the single-particle relations

$$\begin{aligned} \phi(u, m) &= g(v_{m+1} - v_m)\phi'(u, m) + f(v_{m+1} - v_m)\phi'(u, m+1) \\ \phi(u, m+1) &= f(v_{m+1} - v_m)\phi'(u, m) + g(v_{m+1} - v_m)\phi'(u, m+1) \end{aligned} \tag{3.3}$$

where $\phi'(u, m)$ is obtained from $\phi(u, m)$ by interchanging v_m with v_{m+1}. The fourth of the relations (2.8), with $Z(X \mid V)$ replaced by $f(X \mid V \mid U)$, follows from the fact that

$$J = \sinh(u-u'+\lambda)\phi(u, m)\phi(u', m+1) - \sinh(u'-u+\lambda)\phi(u', m)\phi(u, m+1) \tag{3.4}$$

is a symmetric function of v_m and v_{m+1}. This can be established by noting that

$$J = \sinh(u-u'+\lambda)\sinh(v_{m+1}-u)\sinh(u'-v_m+\lambda)$$
$$-\sinh(u'-u+\lambda)\sinh(v_{m+1}-u')\sinh(u-v_m+\lambda) \tag{3.5}$$

(apart from factors independent of v_m and v_{m+1}). Some elementary algebra reveals that (3.5) is symmetric in v_m, v_{m+1}.

So far U has been arbitrary: from now on we require that

$$U = V_l \tag{3.6}$$

i.e. U is the set of n left-rapidities.

The RHS of (3.2) is a symmetric function of $u_1, \ldots, u_n$, so the ordering of the set V_l in (3.6) is irrelevant and the choice of U is unaffected by the fact that v_m and v_{m+1} are interchanged in (2.8). Since m and $m+1$ in (2.8) cannot be end-points of the same line, it is also true that the division of V into V_l and V_r is unchanged by using (2.8).

3.2. End condition

Now consider the basic graph of figure 5, where

$$V_l = \{v_1, \ldots, v_n\} \tag{3.7}$$

and $v_{n+1}, \ldots, v_{2n}$ can be regarded as given by (2.12*a*). Since the sum in (3.2) is over permutations, it follows that in the summand $u_1, \ldots, u_n$ is a permutation of $v_1, \ldots, v_n$, i.e.

$$\{u_1, \ldots, u_n\} = P\{v_1, \ldots, v_n\}. \tag{3.8}$$

First consider the various arrow configurations at end-points n and $n+1$, i.e. the end-points of the right-hand line in figure 5. If the arrows at these points both point in, no x_r in (3.2) is equal to n or $n+1$. From (3.1) it follows that $\phi(u_r, x_r)$ contains a factor $\sinh(u_r - v_n)$, so the summand in (3.2) contains a factor

$$\prod_{k=1}^{n} \sinh(u_k - v_n). \tag{3.9}$$

From (3.8), one of $u_1, \ldots, u_n$ is equal to v_n, so (3.9), and hence $f(X \mid V \mid U)$, vanishes.

If both arrows point out, so that $x_r = n$, $x_{r+1} = n+1$ for some value of r, then the summand of (3.2) still contains the factor (3.9), but without the $k = r$, $r+1$ terms. By grouping together the terms obtained by interchanging u_r with u_{r+1}, we also get the factor (3.4), and hence (3.5), with $u = u_r$, $u' = u_{r+1}$, $m = n$. Using $v_{n+1} = v_n + \lambda$, the factor (3.5) now simplifies to

$$2\cosh\lambda \sinh(u_r - v_n)\sinh(u_{r+1} - v_n)\sinh(u_{r+1} - u_r). \tag{3.10}$$

Thus this contributes the $k = r$, $r+1$ terms and again we get the full factor (3.9), so that $f(X \mid V \mid U)$ vanishes.

If one arrow points in and one out (at end-points n and $n+1$), then either $x_r = n$, $x_{r+1} > n+1$, or $x_{r-1} < n$, $x_r = n+1$. In both cases we get a factor (3.9), but missing the $k = r$ term. Thus the summand of (3.2) vanishes unless $u_r = v_n$, so we need only look at such terms.

Consider all the factors in the summand (3.2) that involve u_r, v_n or v_{n+1}. The double (i, j) product gives a contribution

$$\prod_{k=1}^{r-1} \frac{\sinh(u_k - u_r + \lambda)}{\sinh(u_r - u_k)} \times \prod_{k=r+1}^{n} \frac{\sinh(u_r - u_k + \lambda)}{\sinh(u_k - u_r)}. \tag{3.11a}$$

The function $\phi(u_r, n)$, or $\phi(u_r, n+1)$, gives

$$\sinh\lambda \prod_{j=1}^{n-1} \sinh(u_r - v_j + \lambda) \sinh(\lambda + v_j - u_r), \tag{3.11b}$$

and the $j = n$, $n+1$ factors of the $n-1$ other single particle functions $\phi(u_k, x_k)$ give

$$\prod_{k=1}^{r-1} \sinh(v_n - u_k) \sinh(\lambda + v_n - u_k) \times \prod_{k=r+1}^{n} \sinh(u_k - v_n + \lambda) \sinh(u_k - v_n). \tag{3.11c}$$

Multiplying (3.11a) and (3.11c), remembering that $u_r = v_n$, we get a combined factor

$$\prod_{\substack{k=1 \\ \neq r}}^{n} \sinh(\lambda + v_n - u_k) \sinh(\lambda - v_n + u_k). \tag{3.12}$$

Since $u_r = v_n$ and $u_1, \ldots, u_{r-1}, u_{r+1}, \ldots, u_n$ is a permutation of $v_1, \ldots, v_{n-1}$, (3.12) is the same as the product in (3.11b). Altogether we obtain the factor

$$R = \sinh\lambda \prod_{\substack{j=1 \\ \neq r}}^{n} [\sinh(\lambda + u_r - u_j) \sinh(\lambda - u_r + u_j)]^2. \tag{3.13}$$

This multiplies all non-zero terms in the summand of (3.2). Removing it, noting that the sum is now over $(n-1)!$ permutations of $v_1, \ldots, v_{n-1}$, we obtain the original expression for $f(X|V_0|U)$, but with n replaced by $n-1$, $v_1, \ldots, v_n$ by $v_1, \ldots, v_{n-1}$, and each x_j constrained to the values $1, \ldots, n-1$, $n+2, \ldots, 2n$ (similarly for j in (3.1)). This is just the function $f(X|V_0|U)$ for the graph in figure 5, with the right-hand line removed: let us write it as $[f(X|V_0|U)]_{\text{reduced}}$.

Thus for the graph in figure 5, if there is one in arrow and one out arrow at the ends of the right-hand line, then

$$f(X|V_0|U) = R[f(X|V_0|U)]_{\text{reduced}} \tag{3.14a}$$

otherwise

$$f(X|V_0|U) = 0. \tag{3.14b}$$

We can now proceed by induction to establish that

$$\begin{aligned} f(X|V_0|U) &= C \qquad \text{if } \{y_1, \ldots, y_n\} \text{ is a permutation of } \{1, \ldots, n\}, \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{3.15}$$

where

$$C = RC_{\text{reduced}}$$
$$= (\sinh\lambda)^n \prod_{\substack{1\leqslant i,j\leqslant n \\ i\neq j}} \sinh^2(\lambda + u_i - u_j) \tag{3.16}$$

and $y_r = \min(x_r, 2n+1-x_r)$, as in (2.12*b*).

3.3. Expression for Z(X\|V)

Thus the function $C^{-1}f(X|V|U)$, with $U = V_l$, satisfies both the recursion relation (2.8) and the 'end-condition' (2.12). It follows that

$$Z(X\,|\,V) = C^{-1}f(X|V|V_l), \tag{3.17}$$

i.e. the partition function of any Z-invariant six-vertex model is given by the Bethe ansatz expression (3.2) (divided by C), with $u_1, \ldots, u_n$ set to the left-rapidities. If no two lines of $\mathscr{L}$ are in the disjoint configuration 2(*b*), then $u_1, \ldots, u_n = v_1, \ldots, v_n$.

Note that the Bethe ansatz expression (3.2) is the same (apart from a normalisation) as that for the eigenvectors of the transfer matrix of the square lattice Z-invariant inhomogeneous six-vertex model (equations (79) and (80) of Baxter 1973) with 2η, u_j, $w_j+\eta$, N replaced by $i\lambda$, iu_j, iv_j, $2n$). In that case, $u_1, \ldots, u_n$ are given by, for $i = 1, \ldots, n$,

$$\prod_{j=1}^{2n} \frac{\sinh(u_i - v_j + \lambda)}{\sinh(v_j - u_i)} = -h^2 \prod_{j=1}^{n} \frac{\sinh(u_i - u_j + \lambda)}{\sinh(u_i - u_j - \lambda)}. \tag{3.18}$$

This condition arises from cylindrical boundary conditions (h is a boundary field term) which do not apply here. Even so, since u_i is equal to some left-rapidity v_k, with corresponding right-rapidity $v_l = v_k + \lambda$, the LHS factors $\sinh(v_k - u_i)$, $\sinh(u_i - v_l + \lambda)$ vanish. The LHS therefore becomes 0/0, and (3.18) can be said to be satisfied by (3.6), for $i = 1, \ldots, n$.

The Bethe ansatz can also be applied to the diagonal-to-diagonal transfer matrix (Kelland 1974), or (by using helical boundary conditions) to the single-vertex transfer matrix (Baxter 1969). For a solvable six-vertex model, one always finds that the eigenvector has the form (3.2) (possibly with some modification of (3.18)). This is because (3.2) is a general solution of the local vertex recurrence relation (2.8).

To summarise: the partition function $Z(X\,|\,V)$ of any Z-invariant six-vertex model is given by (3.17), (3.16), (3.2) and (3.1), i.e. it is given by a Bethe ansatz. Unlike the usual applications of the Bethe ansatz, it is not necessary to solve the set of equations (3.18) for $u_1, \ldots, u_n$; instead they are simply some permutation of the left-rapidities. (In fact this is a solution of (3.18).)

If, as is often the case, no two lines of $\mathscr{L}$ are in the disjoint configuration 2(*b*), then we can take $u_1, \ldots, u_n = v_1, \ldots, v_n$.

4. Inhomogeneous square lattice

We can specialise our result to the square lattice of r rows and s columns, shown in figure 6. Let $v_1, \ldots, v_r$ be the rapidities of the left-hand ends of the horizontal lines, and $w_1, \ldots, w_s$ the rapidities of the lower ends of the vertical lines. Take B to be at

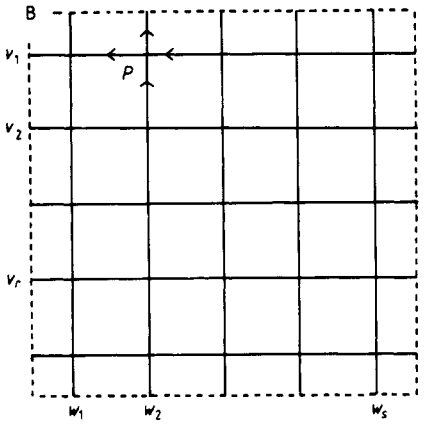

Figure 6. The square lattice of r rows and s columns, with left-rapidities $v_1, \ldots, v_r$, $w_1, \ldots, w_s$.

the top-left corner, as indicated. Then $v_1, \ldots, v_r, w_1, \ldots, w_s$ are all 'left-rapidities' in the sense of §§ 2 and 3. The Boltzmann weights are given by (2.3), so for instance the weight of the arrow configuration shown at site P is $f(w_2 - v_1)$.

The full set of rapidities used in (3.1) is $v_1, \ldots, v_{2n}$, where

$$\begin{aligned} &n = r + s \\ &v_{r+1}, \ldots, v_n = w_1, \ldots, w_s \\ &v_{n+1}, \ldots, v_{n+r} = v_r + \lambda, \ldots, v_1 + \lambda \\ &v_{n+r+1}, \ldots, v_{2n} = w_s + \lambda, \ldots, w_1 + \lambda. \end{aligned} \tag{4.1}$$

The indexes of the sequences on the RHS of the last two equations are in decreasing order; $v_1, \ldots, v_r$ have the same meaning here as in the previous sections.

The parameters $u_1, \ldots, u_n$ in (3.2) are the 'left-rapidities', so we can take

$$\begin{aligned} &u_1, \ldots, u_r = v_1, \ldots, v_r \\ &u_{r+1}, \ldots, u_n = w_1, \ldots, w_s. \end{aligned} \tag{4.2}$$

The partition function $Z(X \mid V)$ is now given by (3.17), (3.16), (3.2) and (3.1). As was remarked in § 1, (3.2) is a sum of $n!$ terms, so is still a long way from being immediately useful. Also, whenever any two of $u_1, \ldots, u_n$ become equal, the sum (3.2) contains infinite terms, but they add together in pairs to give a finite total. For the homogeneous six-vertex model (which is the usual case of interest), we have $v_1 = \ldots = v_r$ and $w_1 = \ldots = w_s$. Thus many of $u_1, \ldots, u_n$ are equal, and a large number of delicate limits would need to be taken in order to use (3.2) as written.

Obviously it would be a great improvement if (3.2) could be cast in a more tractable form, perhaps using ideas similar to those employed by Gaudin *et al* (1981), and Korepin (1982), for calculating the normalisation $\langle \bar{f} \mid f \rangle$ of transfer matrix eigenvectors.

One idea is to substitute $f(X \mid V \mid V_0)$ into these normalisation expressions: the result can be regarded as the combined partition function of two Z-invariant models M and $\bar{M}$ on $\mathscr{L}$, joined at their boundaries and summed over all allowed boundary arrow arrangements. The 'conjugate' $\bar{f}$ of f is given (p 162 of Baxter 1973) by interchanging the expressions $u - v_j + \lambda$ and $v_j - u$ in (3.1), and replacing $u_i - u_j + \lambda$ by $u_i - u_j - \lambda$ in

(3.2). Each function $\bar{\phi}(u, x)$ then contains factors $\sinh(u - v_k)$, for all left-rapidities v_k. These factors must be removed (corresponding to dividing by $\bar{C}$) before setting $U = V_l$.

Sadly, it appears that the resulting expression for $\langle \bar{f} | f \rangle$ exactly vanishes (even though it is a sum of non-zero terms). This can happen because M and $\bar{M}$ cannot both have only positive weights. (One can also say that the $2n$-column transfer matrix with eigenvector f cannot be Hermitian.) This shows that $f(X|V|V_l)$, with V satisfying (2.1), is a particularly special case of the Bethe ansatz.

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